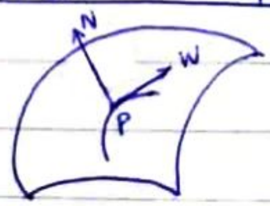
03/12/2018

Κάθετη καμπυλότητα:



$$k_n(w) = \frac{II_p(w)}{I_p(w)}$$

$k_n(w) = k_1(p)\cos^2\theta + k_2(p)\sin^2\theta$

$w = \cos\theta\, e_1(p) + \sin\theta\, e_2(p)$

Κύριες καμπυλότητες:

$k_1(p) = \max\{ k_n(w) \,/\, w \in T_pS,\ \|w\| = 1 \}$

$k_2(p) = \min\{ k_n(w) \,/\, w \in T_pS,\ \|w\| = 1 \}$, $k_1, k_2 : S \to \mathbb{R}$

Υπάρχει ορθοκ. βάση $\{e_1(p), e_2(p)\}$ τ.ω. T_pS ώστε:

$L_p\, e_1(p) = k_1(p)\, e_1(p)$

$L_p\, e_2(p) = k_2(p)\, e_2(p)$

Καμπυλότητα Gauss:

$K(p)$, $k_1, k_2(p) = \det L_p$

Μέση καμπυλότητα: $H(p) = \frac{1}{2}(k_1(p) + k_2(p)) = \frac{1}{2} \operatorname{trace} L_p$

$H^2(p) \geq K(p)$, $H^2(p) = K(p) \Leftrightarrow k_1(p) = k_2(p)$

$K, H : S \to \mathbb{R}$ είναι λείες:

$$K \circ X = \frac{eg - f^2}{EG - F^2} , \quad H \circ X = \frac{Eg - 2Ff + Ge}{2(EG - F^2)}$$

$$k_1 = H + \sqrt{H^2 - K} , \quad k_2 = H - \sqrt{H^2 - K}$$

Επιφάνειες γραφήματα:

$h: U \subset \mathbb{R}^2 \to \mathbb{R}$ λεία

$\Gamma_h = \{(x, y, h(x,y)) \,/\, (x,y) \in U\}$

$X: U \to \Gamma_h$: $X(x,y) = (x, y, h(x,y))$

$$N = \frac{(-h_x, -h_y, 1)}{\sqrt{h_x^2 + h_y^2 + 1}}$$

$X_x = (1, 0, h'(x))$, $X_y = (0, 1, h_y)$

$E = \|X_x\|^2 = 1 + h_x^2$

$F = \langle X_x, X_y \rangle = h_x h_y$

$G = \|X_y\|^2 = 1 + h_y^2$

$X_{xx} = (0, 0, h_{xx})$

$X_{xy} = (0, 0, h_{xy})$

$X_{yy} = (0, 0, h_{yy})$

$$e = \langle X_{xx}, N \rangle = \frac{h_{yy}}{\sqrt{h_x^2 + h_y^2 + 1}}$$

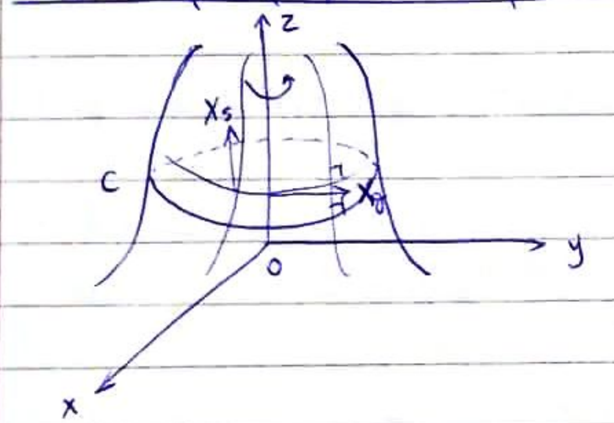
$$f = \langle X_{xy}, N \rangle = \frac{h_{xy}}{\sqrt{\dots}} , \quad g = \langle X_{yy}, N \rangle = \frac{h_{yy}}{\sqrt{\dots}}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1+h_x^2 & h_x h_y \\ h_x h_y & 1+h_y^2 \end{pmatrix}$$

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \frac{1}{\sqrt{h_x^2+h_y^2+1}} \begin{pmatrix} h_{xx} & h_{xy} \\ h_{xy} & h_{yy} \end{pmatrix}$$

$$K = \frac{h_{xx} h_{yy} - h_{xy}^2}{(h_x^2+h_y^2+1)^2} \quad , \quad H = \frac{(1+h_y^2)\, h_{xx} - 2 h_x h_y h_{xy} + (1+h_x^2) h_{yy}}{2(h_x^2+h_y^2+1)^{3/2}}$$

Επ. περιστροφής επιφάνειες:



$c : I \to Oxz$ με παράμετρο το μήκος τόξου και παραμετρική παράσταση:

$c(s) = (\varphi(s), 0, \psi(s))$

$\varphi(s) > 0$

$(\dot\varphi(s))^2 + (\dot\psi(s))^2 = 1 \ , \ \forall\, s \in I$

$X : I \times \mathbb{R} \to \mathbb{R}^3$

$X(s,\theta) = (\varphi(s)\cos\theta, \varphi(s)\sin\theta, \psi(s))$

$$N = \frac{X_s \times X_\theta}{\| X_s \times X_\theta \|} \quad , \quad \begin{aligned} X_s(s,\theta) &= (\dot\varphi(s)\cos\theta, \dot\varphi(s)\sin\theta, \dot\psi(s)) \\ X_\theta(s,\theta) &= (-\varphi(s)\sin\theta, \varphi(s)\cos\theta, 0) \end{aligned}$$

$$X_s \times X_\theta (s,\theta) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \dot\varphi(s)\cos\theta & \dot\varphi(s)\sin\theta & \dot\psi(s) \\ -\varphi(s)\sin\theta & \varphi(s)\cos\theta & 0 \end{vmatrix} =$$

$$= (-\varphi(s)\dot\psi(s)\cos\theta, \ -\varphi(s)\dot\psi(s)\sin\theta, \ \varphi(s)\dot\varphi(s))$$

$$\| X_s \times X_\theta (s,\theta) \| = \sqrt{\varphi^2(s) - (\dot\psi(s))^2 + \varphi^2(s)(\dot\varphi(s))^2} = \varphi(s)$$

$$\boxed{N(s,\theta) = (-\dot\psi(s)\cos\theta, \ -\dot\psi(s)\sin\theta, \ \dot\varphi(s))}$$

$$E = \|X_s\|^2 = (\dot\varphi(s))^2 + (\dot\psi(s))^2 = 1$$
$$F = \langle X_s, X_\theta \rangle = 0$$
$$G = \|X_\theta\|^2 = \varphi^2$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \varphi^2(s) \end{pmatrix}, \quad \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \dot\varphi\ddot\psi - \dot\psi\ddot\varphi & 0 \\ 0 & \varphi\dot\psi \end{pmatrix}$$

$$X_{ss}(s,\theta) = (\ddot\varphi(s)\cos\theta, \ddot\varphi(s)\sin\theta, \ddot\psi(s))$$
$$X_{s\theta}(s,\theta) = (-\dot\varphi(s)\sin\theta, \dot\varphi(s)\cos\theta, 0)$$
$$X_{\theta\theta}(s,\theta) = (-\varphi(s)\cos\theta, -\varphi(s)\sin\theta, 0)$$

$$e = \langle X_{ss}, N \rangle = -\ddot\varphi\dot\psi\cos^2\theta - \ddot\varphi\dot\psi\sin^2\theta + \ddot\psi\dot\varphi$$
$$\boxed{e = \dot\varphi\ddot\psi - \dot\psi\ddot\varphi}$$
$$\boxed{f = \langle X_{s\theta}, N \rangle = 0}$$
$$\boxed{g = \langle X_{\theta\theta}, N \rangle = \varphi\dot\psi}$$

$$(\dot\varphi)^2 + (\dot\psi)^2 = 1$$
$$2(\dot\varphi\ddot\varphi + \dot\psi\ddot\psi) = 0 \Rightarrow \dot\psi\ddot\psi = -\dot\varphi\cdot\ddot\varphi$$

Καμπυλότητα Gauss:

$$K = \frac{eg - f^2}{EG - F^2} = \frac{(\dot\varphi\ddot\psi - \dot\psi\ddot\varphi)\varphi\dot\psi}{\varphi^2} = \frac{\dot\varphi\dot\psi\ddot\psi - (\dot\psi)^2\ddot\varphi}{\varphi} =$$
$$= -\frac{(\dot\varphi)^2\ddot\varphi + (\dot\psi)^2\ddot\varphi}{\varphi} \Longrightarrow \boxed{K = -\frac{\ddot\varphi}{\varphi}}$$

Παραδείγματα επιφανειών με $K=0$:

$$\left.\begin{array}{l} K = 0 \Leftrightarrow \ddot\varphi(s) = 0 \Leftrightarrow \dot\varphi(s) = a \Leftrightarrow \varphi(s) = as + a_0 \\ (\dot\varphi)^2 + (\dot\psi)^2 = 1 \longrightarrow \dot\psi(s) = b \Leftrightarrow \psi(s) = bs + b_0 \end{array}\right\} \Rightarrow$$

$\Rightarrow$ Η c είναι ευθ. τμήμα.

$K = \frac{1}{R^2}$

Ερωτημα: Υπάρχουν άλλες επιφάνειες με καμπυλότητα Gauss 1 εκτός της S^2;

$K = 1 \iff -\frac{\ddot{\varphi}}{\varphi} = 1 \iff \boxed{\ddot{\varphi} + \varphi = 0}$

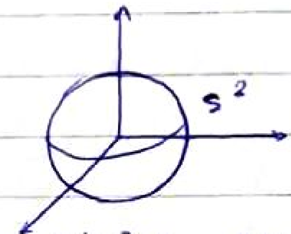
$\boxed{\varphi(s) = a\cos s}$

$(\dot{\varphi}(s))^2 + (\dot{\psi}(s))^2 = 1 \iff a^2\sin^2 s + (\dot{\psi}(s))^2 = 1 \iff$

$\iff \dot{\psi}(s) = \pm\sqrt{1 - a^2\sin^2 s}$

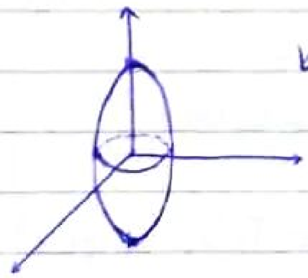
$\psi(s) = \int \sqrt{1 - a^2\sin^2 s}\, ds$

- Αν $a = 1$ $\quad \psi(s) = \sin s$



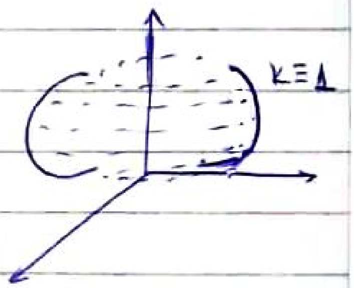
- Αν $a \neq 1$, τότε η c ΔΕΝ είναι κύκλος και επομένως η παραγόμενη ~~επιφάνεια~~ εκ περιστροφής επιφάνεια ΔΕΝ είναι η S^2

- $a < 1$



$K \equiv 1$

- $a > 1$



$N_s(s,\theta) = (-\ddot{\psi}(s)\cos\theta, -\ddot{\psi}(s)\sin\theta, \ddot{\varphi}(s))$

$N_\theta(s,\theta) = (\dot{\psi}(s)\sin\theta, -\dot{\psi}(s)\cos\theta, 0)$

$L X_s = -N_s \quad , \quad L X_\theta = -N_\theta$

Το $\{X_s, \frac{X_\theta}{\|X_\theta\|} = \frac{X_\theta}{\varphi}\}$ είναι ορθοκανονική βάση του εφαπτόμενου επιπέδου

$$\left.\begin{aligned} N_s &= \langle N_s, X_s\rangle X_s + \langle N_s, \tfrac{X_\theta}{\varphi}\rangle \tfrac{X_\theta}{\varphi} \\ N_\theta &= \langle N_\theta, X_s\rangle X_s + \langle N_\theta, \tfrac{X_\theta}{\varphi}\rangle \tfrac{X_\theta}{\varphi} \end{aligned}\right\} \iff$$

$X_s = (\dot{\varphi}\cos\theta, \dot{\varphi}\sin\theta, \dot{\psi})$

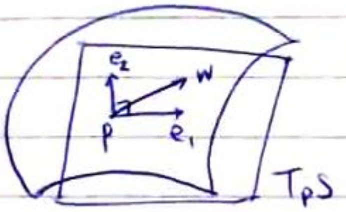
$X_\theta = (-\varphi\sin\theta, \varphi\cos\theta, 0)$

$$\Longleftrightarrow \begin{cases} N_s = -(\dot\varphi\ddot\psi - \dot\psi\ddot\varphi)\, X_s \\ N_\theta = \varphi\cdot\dot\psi\, \dfrac{X_\theta}{\varphi^2} = \dfrac{\dot\psi}{\varphi} X_\theta \end{cases} \Bigg| \Longrightarrow \begin{cases} L X_s = \kappa X_s \\ L X_\theta = \dfrac{\dot\psi}{\varphi} X_\theta \end{cases}$$

↓

κύριες διευθύνσεις

$\kappa, \dfrac{\dot\psi}{\varphi}$: κύριες καμπυλότητες



$\mathrm{II}_p : T_pS \to \mathbb{R}$, $\mathrm{II}_p(w) = \langle L_p w, w\rangle_p$

$D_p = \{ w \in T_pS \, / \, \mathrm{II}_p(w) = \pm 1 \}$

δεικτίουσα Dupin

$L_p e_1 = k_1(p) e_1$

$L_p e_2 = k_2(p) e_2$

$\boxed{w = x e_1 + y e_2}$

$\mathrm{II}_p(w) = \langle L_p w, L_p w\rangle = \langle L_p(x\cdot e_1 + y\cdot e_2), x e_1 + y e_2\rangle =$

$= \langle x L e_1 + y L e_2, x e_1 + y e_2\rangle = \langle x k_1(p) e_1 + y k_2(p) e_2, x e_1 + y e_2\rangle$

$\Longrightarrow \boxed{\mathrm{II}_p(w) = k_1(p)\, x^2 + k_2(p)\, y^2}$

$D_p = \{ w = x e_1 + y e_2 \in T_pS \, / \, k_1(p)\, x^2 + k_2(p)\, y^2 = \pm 1 \}$

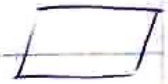
κωνική τομή

- II_p θετική ή αρνητική ορισμένη $\Leftrightarrow k_1(p) k_2(p) > 0 \Leftrightarrow$ $\Leftrightarrow K(p) > 0 \Leftrightarrow D_p$ έλλειψη
- II_p αόριστη $\Leftrightarrow k_1(p) k_2(p) < 0 \Leftrightarrow K(p) < 0 \Leftrightarrow$ $\Leftrightarrow D_p$ υπερβολή
- II_p θετική ή αρνητική ημιορισμένη $\Leftrightarrow k_1 k_2(p) = 0$, $k_1(p) + k_2(p) \neq 0 \Leftrightarrow K(p) = 0 \neq H(p) \Leftrightarrow D_p$ ζεύγη ευθειών

Ταξινόμηση των σημείων:

Ορισμός: Το σημείο p καλείται:

- Ελλειπτικό $\Leftrightarrow K(p) > 0$ π.χ.  S^2_R, $r = \frac{1}{R^2}$
- Υπερβολικό $\Leftrightarrow K(p) < 0$ π.χ.

- Παραβολικό $\Leftrightarrow$ $K(p) = 0 \neq H(p)$ π.χ. H
- Ισόπεδο $\Leftrightarrow$ $K(p) = 0 = H(p) \Leftrightarrow k_1(p) = k_2(p)$ π.χ. 

  $\Leftrightarrow \mathrm{II}_p = 0$.
- Ομφαλικό $\Leftrightarrow$ $k_1(p) = k_2(p) \neq 0 \Leftrightarrow H^2(p) = K(p) > 0$ π.χ.  S^2_R